$$\pi(x) = b^2 \int_{-\infty}^{\infty} \frac{1}{(x-x')^2} \cdot \frac{1}{(x')^2} dx' = -b^2 \delta''(x)$$

Hence, and from (5.4), we finally obtain

$$t(x) = \sigma_0^2 - b^2 \frac{d^2}{dx^2} \left[\frac{1}{l_0} D_\infty \delta(x) + \psi(x) \bar{\sigma}^2(x) \right]$$

A graph of the continuous part of the function $t(x) - \sigma_0^2$ is shown in Fig.3. The presence of a singular component and a singularity at $x = b$ in the correlation function of the random field $\sigma_{nn}(x)$ on the line of defects is due to the replacement of the real cracks by point defects. For a random field of inhomogeneities of finite size the correlation function should be smooth, bounded, and have minimal correlation radius of the order of the mean size of the defect. As a random field of defects approaches a regular lattice, the correlation radius of the stress field grows, as is also seen from Fig.3 (the physically meaningless domain $x \leqslant b$ is not shown in Fig.3).

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NON-AXISYMMETRIC BUCKLING AND POST-CRITICAL BEHAVIOUR OF ELASTIC SPHERICAL SHELLS IN THE CASE OF A DOUBLE CRITICAL VALUE OF THE LOAD*

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The influence of small geometric imperfections of the shape of the middle surface on the non-axisymmetric buckling and initial post-critical behaviour of shallow elastic spherical shells is investigated for a uniform external pressure.

Cases are considered when the least bifurcation load of non-axisymmetric buckling p_0 of the corresponding ideal shell /1/ is a double eigenvalue of the linearized problem, i.e., buckling in two eigen modes occurs. Surfaces of values of the upper critical load as a function of imperfection functionals are constructed by using matrix pivotal condensation /1-7/ and alignment /8-10/ methods for shells with a closed framed edge for $\Lambda = 6.6$ and 9, with a free clamped edge for $\Lambda = 8.045$, and with a fixed hinge-supported edge for $\Lambda = 5.655$ and $\Lambda \to \infty$, where the parameter is $\Lambda = 2[3(1-\nu^2)]^{1/4}(H/h)^{1/2}$, and H is the height of the shell rise, h is its thickness, and ν is Poisson's ratio.

Formulas for the theory of the initial post-critical behaviour of spherical shells with elastic clamping of the edge in a fixed wall have been obtained by the perturbation method /7/, and in an asymptotic analysis as $\Lambda\to\infty$ by the perturbation method in combination with the boundary-layer method. Methodologically this case is interesting because of the presence of non-linearity in the boundary conditions.

The influence of initial imperfections on the non-axisymmetric buckling of spherical shells under linear boundary conditions was investigated in /2,3/, when p_0 is an isolated eigenvalue, and in /4–6/ for $\Lambda\geqslant 17$, when p_0 is a double eigenvalue. An asymptotic analysis as $\Lambda\to\infty$ in the case of a closed framed edge is given in /3,11/.

1. Formulation of the problem. The system of non-linear equilibrium differential equations of shallow elastic spherical shells with small initial imperfections in the middle surface shape /12/ under uniform external pressure can be represented in the form

$$\begin{gathered}\varepsilon_0^2\nabla^4w-\nabla^2F-[w,F]+\xi[\zeta,F]=4p\\ \varepsilon_0^2\nabla^4F+\nabla^2w+\frac{1}{2}[w,w]-\xi[\zeta,w]=0\\ \varepsilon_0=\Lambda^{-1},\quad \nabla^2=(\,)''+\frac{1}{r}(\,)'+\frac{1}{r^2}(\,)^{\cdot\cdot}\\ \nabla^4=\nabla^2\nabla^2,\ [w,F]=w''\left(\frac{1}{r}F'+\frac{1}{r^2}F^{\cdot\cdot}\right)+\\ F''\left(\frac{1}{r}w'+\frac{1}{r^2}w^{\cdot\cdot}\right)-2\left(\frac{1}{r}w'^{\cdot}-\frac{1}{r^2}w^{\cdot}\right)\left(\frac{1}{r}F'^{\cdot}-\frac{1}{r^2}F^{\cdot}\right)\\ (\,)'=\frac{\partial}{\partial r}(\,),\quad (\,)^{\cdot}=\frac{\partial}{\partial\theta}(\,),\quad 0\leqslant r\leqslant 1,\quad 0\leqslant\theta\leqslant 2\pi\end{gathered}\tag{1.1}$$

We will consider these equations together with each of the boundary conditions on the contour $r=1$

$$\begin{gathered}1)\ w=\Gamma_1w=F=F'=0;\quad 2)\ w=\Gamma_1w=\Gamma_2F=0,\\ \varepsilon_0^2\Gamma_3F+\mu_1\left(w'-\xi\zeta'w'+\frac{1}{2}w'^2\right)=0;\quad \Gamma_1w=\mu_1(w''+\nu w'+\\ k\varepsilon_0^{-1}w')+\mu_2w',\quad \Gamma_2F=F''-\nu F'-\nu F^{\cdot\cdot}\\ \Gamma_3F=F'''+(2+\nu)F^{\cdot\cdot\prime}+(\nu-1)F'-3F^{\cdot\cdot},\ 0<\nu<0.5,\ \zeta=0\end{gathered}\tag{1.2}$$

Here $\mu_1=0,\mu_2=1$ or $\mu_1=1,\mu_2=0$, k is the coefficient of elastic clamping of the edge in a fixed wall, $\xi\zeta(r,\theta)$ is the initial deflection, ξ is a scalar parameter, and $|\xi|\ll 1$. The dimensionless quantities in (1.1) and (1.2) are related to the dimensional quantities by the formulas given in /6,13/.

Problems (1.1) and (1.2) for any p and $\xi=0$ have axisymmetric solutions $x^*(r,p)=(w^*(r,p),F^*(r,p))$, which are determined from the appropriate boundary conditions

$$\begin{gathered}\varepsilon_0^2A_0u=uv+rv+2pr^2\\ \varepsilon_0^2A_0v=-\frac{1}{2}u^2-ru\\ A_0u=r(r^{-1}(ru)')',\ |ur^{-1},vr^{-1}|_{r=0}<\infty\\ w^*(1,p)=0,\ u(r,p)=w^{*\prime},\ v(r,p)=F^{*\prime}\\ 1)\ \mu_1(u'+\nu u+k\varepsilon_0^{-1}u)+\mu_2u=v=0,\ r=1\\ 2)\ \mu_1(u'+\nu u+k\varepsilon_0^{-1}u)+\mu_2u=v'-\nu v=0,\ r=1\end{gathered}\tag{1.3}$$

It is shown in /1,10/ that critical loads $p=p_0$ exist for which non-axisymmetric modes can exist together with x^*. These loads are defined as eigenvalues of the boundary value problems

$$\begin{gathered}l_n^{(1)}x_n\equiv\varepsilon_0^2L_n^2w_n-L_nf_n-\frac{u_0}{r}f_n''-\frac{u_0'}{r}\left(f_n'-\frac{n^2}{r}f_n\right)-\\ \frac{v_0'}{r}\left(w_n'-\frac{n^2}{r}w_n\right)-\frac{v_0}{r}w_n''=0\\ l_n^{(2)}x_n\equiv-\varepsilon_0^2L_n^2f_n-L_nw_n-\frac{u_0}{r}w_n''-\frac{u_0'}{r}\left(w_n'-\frac{n^2}{r}w_n\right)=0,\\ x_n=(w_n,f_n)\\ L_n(\,)=(\,)''+\frac{1}{r}(\,)'-\frac{n^2}{r^2}(\,),\quad f_n=O(r^n),\quad w_n=O(r^n)\\ u_0=u(r,p_0),\ v_0=v(r,p_0);\ n=1,2,\ldots\\ 1)\ w_n=\Gamma_1w_n=f_n=f_n'=0,\ r=1\\ 2)\ w_n=\Gamma_1w_n=R_2(n,f_n)=R(n,x_n)=0,\ r=1\\ R_2(n,f)=f''-\nu f'+\nu n^2f;\ R(n,x_n)=\varepsilon_0^2[f_n'''-(2+\nu)n^2f_n'+\\ 3n^2f_n+(\nu-1)f_n']+\mu_1(1+u_0)w_n'\end{gathered}\tag{1.4}$$

Problems (1.4) are obtained by linearizing (1.1) and (1.2) with respect to $x^*(r)$, where the eigenvector-functions are found in the form $x_n(r)\cos n\theta$, where n is an integer. Note that the values of p_0 are calculated in /1,10,13,14/ for different Λ and n.

2. Application of the perturbation method. The influence of small imperfections will be investigated by the perturbation method /7/. Setting $w = w^* + \omega$, $F = F^* + \psi$, $p = p_0 + \lambda$, for the small perturbations λ, $x = (\omega, \psi)$ we obtain boundary value problems from (1.1) and (1.2) which we write in vector form

$$M_0x = \Big([\omega, \psi] + \sum_{s=1} \lambda^s([w_s^*, \psi] + [F_s^*, \omega]) - \xi\sum_{m=0} \lambda^m[\zeta, F_m^*] - \xi[\zeta, \psi], \frac{[\omega, \omega]}{2} + \sum_{s=1}\lambda^s[w_s^*, \omega] - \xi\sum_{m=0}\lambda^m[w_m^*, \zeta] - \xi[\zeta, \omega]\Big) \tag{2.1}$$

$$M_0x \equiv (\varepsilon_0^2\nabla^4\omega - \nabla^2\psi - [w^*(p_0), \psi] - [F^*(p_0), \omega], -\varepsilon_0^2\nabla^4\psi - \nabla^2\omega - [w^*(p_0), \omega]),\ \{w_m^*, F_m^*\} = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\{w^*, F^*\}|_{p=p_0}$$

$$1)\ \omega = \Gamma_1\omega = \psi = \psi' = 0,\ r = 1;$$

$$2)\ \omega = \Gamma_1\omega = \Gamma_2\psi = 0,\quad \Gamma_0x = \mu_1\Big[-\frac{1}{2}\omega'^2 - (\omega' - \xi\zeta')\sum_{m=1}\lambda^m w_m^{*\prime} + \xi\zeta'(\omega' + u_0)\Big],\quad r = 1$$

$$\Gamma_0x \equiv \varepsilon_0^2\Gamma_3\psi + \mu_1(1 + u_0)\omega'$$

We will seek the solution of (2.1) in the form of the series

$$x = \eta U_1 + \eta^2U_2 + \ldots,\ U_i = (\Omega_i, \Psi_i),\ |\eta| \ll 1 \tag{2.2}$$
$$\lambda = \eta\lambda_1 + \eta^2\lambda_2 + \ldots,\ \xi = \eta^2\xi_2 + \eta^3\xi_3 + \ldots$$

Here η is a small parameter. Substituting (2.2) into (2.1) and equating coefficients of η^1 to zero, we obtain a boundary value problem in the eigenvalues

$$M_0U_1 = 0,\ U_1 = (\Omega_1, \Psi_1) \tag{2.3}$$
$$1)\ \Omega_1 = \Gamma_1\Omega_1 = \Psi_1 = \Psi_1' = 0,\ r = 1$$
$$2)\ \Omega_1 = \Gamma_1\Omega_1 = \Gamma_2\Psi_1 = \Gamma_0U_1 = 0,\ r = 1$$

Seeking the solution of problems (2.3) in the form $U_1 = x_n(r)\cos n\theta$, we arrive at system (1.4).

In /6/ it follows from (4.9)—(4.18) that the boundary value problems (2.3) are self-adjoint. Let p_0 be an eigenvalue of problems (2.3) and let n eigenvector-functions $\varphi_i = (\omega_i, \psi_i)$ satisfying the orthonormality conditions

$$\langle\varphi_i, \varphi_j\rangle \equiv \int_0^{2\pi}\int_0^1(\omega_i\omega_j + \psi_i\psi_j)\,r\,dr\,d\theta = \delta_{ij}$$

correspond to it.

Here δ_{ij} is the Kronecker delta and $\langle\cdot, \cdot\rangle$ is the scalar product in the Hilbert space E^2 of two-dimensional vector functions with square-summable components.

Equating the coefficient of η^2 to zero, we obtain

$$M_0U_2 = ([\Omega_1, \Psi_1] + \lambda_1[w_1^*, \Psi_1] + \lambda_1[F_1^*, \Omega_1] - \xi_2[\zeta, F_0^*],\ {}^1/_2[\Omega_1, \Omega_1] + \lambda_1[w_1^*, \Omega_1] - \xi_2[\zeta, w_0^*]) \tag{2.4}$$

$$(\Omega_1, \Psi_1) = \sum_{i=1}^{n}\alpha_i(\omega_i, \psi_i),\quad U_2 = (\Omega_2, \Psi_2)$$

$$1)\ \Omega_2 = \Gamma_1\Omega_2 = \Psi_2 = \Psi_2' = 0,\ r = 1;$$
$$2)\ \Omega_2 = \Gamma_1\Omega_2 = \Gamma_2\Psi_2 = \Gamma_0U_2 + \mu_1[{}^1/_2\Omega_1'^2 + \lambda_1w_1^{*\prime}\Omega_1' - \xi_2\zeta'u_0] = 0,\ r = 1$$

The real numbers α_i are determined from the conditions for problems (2.4) to be solvable. By using integration by parts and taking account of Lemma 4.1 in /6/, we write these conditions in the form of a system of algebraic equations

$$\sum_{k=1}^{n}\sum_{m=1}^{n}\alpha_k\alpha_m\{(\omega_k,\psi_m,\omega_j)+{}^1/_2(\omega_k,\psi_j,\omega_m)\}+\lambda_1T_{1j}-\xi_2T_{2j}=0 \tag{2.5}$$

$$T_{1j}=\sum_{i=1}^{n}\alpha_i\{(w_1^*,\psi_i,\omega_j)+(F_1^*,\omega_i,\omega_j)+(w_1^*,\psi_j,\omega_i)\},$$

$$T_{2j}=\{(\omega_j,F_0^*,\zeta)+(\psi_j,w_0^*,\zeta)\},\quad (j=1,2,\ldots,n)$$

$$(u,v,w)=\int_0^{2\pi}\int_0^1[u,v]ur\,dr\,d\theta$$

If the double sum in (2.5) equals zero for all j, and $T_{1k}T_{2m}-T_{1m}T_{2k}\neq 0$ for some pair of subscripts k,m, then $\lambda_1=\xi_2=0$. Then equating the coefficient of η^3 to zero, we obtain an equation for U_3 whose solvability conditions yield a system of algebraic equations for finding the real numbers α_i

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{m=1}^{n}\alpha_i\alpha_k\alpha_m\{(\omega_i,\overset{\bullet}{\psi}_{km},\omega_j)+(\overset{\circ}{\omega}_{km},\psi_i,\omega_j)+(\overset{\circ}{\omega}_{km},\psi_j,\omega_i)\}+ \tag{2.6}$$

$$\lambda_2T_{1j}-\xi_3T_{2j}=0\quad (j=1,2,\ldots,n)$$

$$M_0y_{ml}=([\omega_m,\psi_l]+[\omega_l,\psi_m],[\omega_m,\omega_l])$$

$$y_{ml}=(\overset{\circ}{\omega}_{ml}+\overset{\circ}{\omega}_{lm},\quad \overset{\bullet}{\psi}_{ml}+\overset{\bullet}{\psi}_{lm}),\quad m\neq l$$

$$M_0y_{mm}=([\omega_m,\psi_m],{}^1/_2[\omega_m,\omega_m]),\quad y_{mm}=(\overset{\circ}{\omega}_{mm},\overset{\circ}{\psi}_{mm})$$

Setting $\alpha_i=\mu_i\eta^{-1}$, $\xi_k\approx\xi\eta^{-k}$, $\lambda_k\approx\lambda\eta^{-k}$ $(k=2,3)$, we obtain from (2.5) and (2.6), respectively, a system of bifurcation equations of the form (5) in /4/, obtained for the linear boundary conditions.

Let two eigenvector-functions $\varphi_1(r,\theta)=x_s(r)\cos s\theta$ and $\varphi_2(r,\theta)=x_m(r)\cos m\theta$ correspond to the eigenvalue p_0 of the problem (2.1), where $m>s>1$, $2s\neq m$, $3s\neq m$, and s and m are integers. Then system (2.6) can be written in the form

$$\Phi_1\equiv\nu_1(a_1\nu_1^2+b_1\nu_2^2+\lambda)+\xi d_1=0 \tag{2.7}$$

$$\Phi_2\equiv\nu_2(a_2\nu_1^2+b_2\nu_2^2+\lambda)+\xi d_2=0$$

The coefficients of the system (2.7) are calculated from the formulas /5/

$$a_1=e_1^{-1}\int_0^1\{0{,}5r(h_1B_1+h_2B_2)+\alpha_1g_2-\beta_1g_1\}\,dr, \tag{2.8}$$

$$b_2=e_2^{-1}\int_0^1\{0{,}5r(H_1D_1+H_2D_2)+\alpha_2G_2-\beta_2G_1\}\,dr$$

$$b_1=e_1^{-1}T,\quad a_2=e_2^{-1}T,\quad e_j=2\int_0^1\beta_jr^2\,dr$$

$$T=\frac{1}{8}\left\{I-8\int_0^1(\beta_2g_1-\alpha_2g_2)\,dr\right\}$$

$$I=\int_0^1\{E_1(I_1-I_2)+E_2(I_3-I_4)+F_1(I_1+I_2)+F_2(I_3+I_4)\}r\,dr$$

$$d_j=(4\pi e_j)^{-1}\int_0^{2\pi}\int_0^1\zeta\cos m_j\theta[-(\gamma_j'v_0)'-(\delta_j'u_0)'+$$

$$r^{-1}m_j^2(v_0'\gamma_j+u_0'\delta_j)]\,dr\,d\theta$$

$$j=1;2,\quad m_1=s,\quad m_2=m$$

We introduce the notation

$$\Delta_1=b_2-b_1,\ \Delta_2=a_1-a_2,\ \Delta_3=a_1b_2-a_2b_1$$

Theorem 2.1. Let the shell have the initial deflection $\xi\zeta_2(r)\cos m\theta$ and let p_0 be a double eigenvalue of problem (1.4). Then problems (1.1) and (1.2) have three limit points $p_s^{(i)}$ $(i=1,2,3)$ in the left semicircle of p_0 in the plane $d_1=0$ if the inequalities $b_1>0$, $b_2>0$, $\Delta_3\Delta_2^{-1}>0$ are satisfied simultaneously, two limit points if any two of these inequalities are satisfied, and one limit point if one of these inequalities is satisfied. Here the $p_s^{(i)}$ are determined from the equations

$$(p_0-p_s^{(1)})^{3/2}={}^3/_2\,|d_2\xi|\,(3b_2)^{1/2},\quad b_2>0 \tag{2.9}$$

$$[b_1^{-1}(p_0-p_s^{(2)})]^{3/2}=|d_2\xi(b_1-b_2)^{-1}|,\quad b_1>0$$

$$(p_0 - p_s^{(3)})^{3/2} = {}^3/_2\,|a_1\Delta_2^{-1} d_2\xi|\,(3\Delta_3\Delta_2^{-1})^{1/2}$$
$$\Delta_3\Delta_2^{-1} > 0, \quad |\xi| \ll 1$$

We have for the corresponding solutions of system (2.7) with $d_1 = 0,\ d_2 \neq 0$ and $L_i = p_0 - p_s^{(i)} > 0$

$$\begin{aligned}
&\nu_1^{(1)} = 0, \quad \nu_2^{(1)} = \pm[{}^1/_3\, L_1 b_2^{-1}]^{1/2};\\
&\nu_1^{(2)} = 0, \quad \nu_2^{(2)} = \pm[L_2 b_1^{-1}]^{1/2}\\
&\nu_1^{(3)} = [(L_3 - b_1 y^2)\, a_1^{-1}]^{1/2}, \quad \nu_2^{(3)} \equiv y = \pm[{}^1/_3\, L_3\Delta_2\Delta_3^{-1}]^{1/2},\\
&\nu_1^{(4)} = -\nu_1^{(3)}\\
&\nu_2^{(4)} = \nu_2^{(3)}, \quad (L_3 - b_1 y^2)\, a_1^{-1} \geqslant 0
\end{aligned} \tag{2.10}$$

The upper (lower sign in (2.10) is taken in cases when the appropriate expression under the modulus symbol in (2.9) is positive (negative). This theorem supplements the similar Theorems 7.2 and 7.3 in /6/.

The boundary value problems to determine the functions in (2.7) are written only for boundary conditions 2) in (1.2) since they are known /5,6/ in the case of linear boundary conditions. The vector-functions $E = (E_1, E_2)$ and $F = (F_1, F_2)$ are found from the problems

$$\begin{aligned}
&l_j^{(1)}E = {}^1/_2(I_1 - I_2), \quad l_j^{(2)}E = \frac{1}{2}(I_3 - I_4)\\
&j = m - s, \quad E_i = O(r^j)\\
&E_1 = \Gamma_1 E_1 = R_2(j, E_2) = R(j, E) + \frac{\mu_1}{4}\gamma_1'\gamma_2' = 0, \ r = 1\\
&l_t^{(1)}F = \frac{1}{2}(I_1 + I_2), \quad l_t^{(2)}F = \frac{1}{2}(I_3 + I_4)\\
&t = m + s, \quad F_i = O(r^t)\\
&F_1 = \Gamma_1 F_1 = R_2(t, F_2) = R(t, F) + \frac{\mu_1}{4}\gamma_1'\gamma_2' = 0, \quad r = 1 \quad (i = 1, 2)\\
&x_s(r) = (\gamma_1, \delta_1), \ x_m(r) = (\gamma_2, \delta_2)
\end{aligned} \tag{2.11}$$

For $m - s = 1$ the first system in (2.11) is converted to the form

$$\begin{aligned}
&\varepsilon_0^2[x_0'' + 3(x_0 r^{-1})'] - (u_0 + r)\, y_0 r^{-1} - v_0 x_0 r^{-1} = V_1 r^{-3}\\
&\varepsilon_0^2[y_0'' + 3(y_0 r^{-1})'] + (u_0 + r)\, x_0 r^{-1} =\\
&\quad -V_2 r^{-3}, \ |x_0 r^{-1}, y_0 r^{-1}|_{r=0} < \infty\\
&\mu_1[x_0' + (2 + \nu + k\varepsilon_0^{-1})\, x_0] + \mu_2 x_0 = y_0' + (2 - \nu)\, y_0 =\\
&\quad 0, \ r = 1, \ x_0 = (E_1 r^{-2})', \ y_0 = (E_2 r^{-2})'
\end{aligned} \tag{2.12}$$

The vector functions $\sigma_1 = (-\beta_1, \alpha_1)$ and $B = (B_1, B_2)$ are found from the boundary value problems

$$\begin{aligned}
&\varepsilon_0^2 A_0\beta_1 = v_0\beta_1 - (u_0 + r)\,\alpha_1 + g_1(r), \ \varepsilon_0^2 A_0\alpha_1 = u_0\beta_1 +\\
&\quad r\beta_1 + g_2(r), \ [\mu_1(\beta_1' + \nu\beta_2 + k\varepsilon_0^{-1}\beta_1) + \mu_2\beta_1 = \alpha_1' - \nu\alpha_1]_{r=1} = 0\\
&\alpha_1(0) = \beta_1(0) = 0
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&l_{2s}^{(1)}B = h_1(r), \quad l_{2s}^{(2)}B = h_2(r), \quad B_i = O(r^{2s})\\
&2rh_1 = [\gamma_1, \delta_1, s] + [\delta_1, \gamma_1, s] + 2s^2[\delta_1][\gamma_1]\, r^{-1}, \quad 2rh_2 = [\gamma_1, \gamma_1, s] +\\
&\quad s^2[\gamma_1][\gamma_1]\, r^{-1}, \quad [\gamma, \delta, m] = \gamma''(\delta' - m^2\delta r^{-1}), \quad [\gamma] = \gamma' - \gamma r^{-1};\\
&B_1 = \Gamma_1 B_1 = R_2(2s, B_2) = R(2s, B) + \frac{\mu_1}{4}\gamma_1'^2 = 0, \quad r = 1
\end{aligned} \tag{2.14}$$

See /5/ for the remaining notation.

3. Application of the method of alignment. The coefficients and right sides of (1.3), (1.4), (2.11)−(2.14) have singularities as $r \to 0$, and consequently difficulties associated with the approximation of the equations in the neighbourhood of the point $r = 0$ occur on integrating the appropriate boundary conditions. By using the change of variables, new modes of writing these boundary value problems can be obtained which are convenient for the effective application of the method of alignment /8−10,15/ (*).

Taking account of the results of Sect.9 in /10/, we assume in (1.4)

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*) The detailed content of Sect.3, as well as an investigation of the buckling of imperfect arbitrary shallow shells by the perturbation method. See also, Srubshchik L.S., Application of the method of alignment to analyze non-axisymmetric buckling and post-critical behaviour of elastic spherical shells. Rostov-on-Don, 1982, 48 p. Deposited in VINITI, 11, 11, No. 5568-82.

$$\{w_n, f_n, L_n w_n, L_n f_n\} = r^{n-1}\{\omega_n{}^\circ, \psi_n{}^\circ, z_n, q_n\} \tag{3.1}$$

Then, to determine p_0 and its eigenvector-functions $x_n{}^\circ = (\omega_n{}^\circ, \psi_n{}^\circ)$ we find

$$\begin{aligned}
&\varepsilon_0{}^2 K_n z_n - (1 + u_0 r^{-1}) q_n - v_0 r^{-1} z_n - (u_0 r^{-1})' T_n \psi_n{}^\circ - \\
&\quad (v_0 r^{-1})' T_n \omega_n{}^\circ = 0 \\
&\varepsilon_0{}^2 K_n q_n + (1 + u_0 r^{-1}) z_n + (u_0 r^{-1})' T_n \omega_n{}^\circ = 0, \ z_n = K_n \omega_n{}^\circ \\
&q_n = K_n \psi_n{}^\circ, \ K_n z = z'' + (2n - 1)(zr^{-1})' \\
&T_n z = z' - (n^2 - n + 1) z r^{-1} \ (n \geqslant 2) \\
&|(\omega_n{}^\circ, \ \psi_n{}^\circ, \ z_n, \ q_n) r^{-1}|_{r=0} < \infty \\
&1)\ \omega_n{}^\circ = \varphi_0 = \psi_n{}^\circ = \psi_n{}^\circ = 0, \ r = 1 \\
&2)\ \omega_n{}^\circ = \varphi_0 = q_n - (1 + \nu) T_n \psi_n{}^\circ = 0 \\
&\varepsilon_0{}^2 \{q_n' + (n - 2) q_n + (1 + \nu)(1 - n^2)[\psi_n{}^{\circ\prime} + (n - 1)\psi_n{}^\circ]\} + \\
&\quad \mu_1 (1 + u_0) \omega_n{}^{\circ\prime} = 0, \ r = 1 \\
&\varphi_0 = \mu_1 [z_n + (\nu - 1 + k\varepsilon_0{}^{-1}) \omega_n{}^{\circ\prime}] + \mu_2 \omega_n{}^{\circ\prime}
\end{aligned} \tag{3.2}$$

We make the change of variables

$$\begin{aligned}
&Y_0 = \{y_1, y_2, y_3, y_4\} = \{-u_0, -r^{-1}(ru_0)', \\
&v_0, r^{-1}(rv_0)'\}, \ Y_1 = \{y_5, y_6, \ldots, y_{12}\} = \\
&\{\omega_n{}^\circ, \omega_n{}^{\circ\prime}, z_n, z_n', \psi_n{}^\circ, \psi_n{}^{\circ\prime}, q_n, q_n'\}
\end{aligned} \tag{3.3}$$

We write problem (1.3) in the form of a system with boundary conditions

$$\begin{aligned}
&Y_0' = \left\{y_2 - \frac{y_1}{r}, \ \varepsilon_0^{-2}\left(-y_3 + \frac{y_1 y_3}{r} - 2pr\right),\right. \\
&\left. y_4 - \frac{y_3}{r}, \ \ \varepsilon_0^{-2}\left(y_1 - \frac{1}{2r} y_1{}^2\right)\right\}, \ \ \delta_0 \leqslant r \leqslant 1 \\
&Y_0' = \{t_0, 0, s_0, 0\}, \ \ 0 \leqslant r \leqslant \delta_0 \ \ (\delta_0 \approx 10^{-3}) \\
&y_1(0) = y_3(0) = 0, \ y_2(0) = 2s_0, \ y_4(0) = 2t_0 \\
&1)\ \mu_1 [y_2 + (\nu - 1 + k\varepsilon_0{}^{-1}) y_1] + \mu_2 y_1 = y_3 = 0, \ \ r = 1 \\
&2)\ \mu_1 [y_2 + (\nu - 1 + k\varepsilon_0{}^{-1}) y_1] + \mu_2 y_1 = y_4 - (1 + \nu) y_3 = 0, \quad r = 1
\end{aligned} \tag{3.4}$$

To determine p_0 and $x_n{}^\circ$ we have the following system from (3.2)

$$\begin{aligned}
&Y_1' = \{y_6, y_7 + S_n y_5, y_8, \ S_n y_7 + \varepsilon_0{}^{-2}(a y_{11} + y_3 y_7 r^{-1} - \\
&\quad b r^{-2} G_n y_9 + c r^{-2} G_n y_5), \ y_{10}, \ y_{11} + S_n y_9, \ y_{12}, \\
&S_n y_{11} + \varepsilon_0{}^{-2}(b r^{-2} G_n y_5 - a y_7)\}, \ \ \delta_0 \leqslant r \leqslant 1 \\
&Y_1' = \{s_1, 0, s_2, 0, s_3, 0, s_4, 0\}, \ \ 0 \leqslant r \leqslant \delta_0 \\
&a = 1 - y_1 r^{-1}, \ b = r y_2 - 2y_1, \ \ c = r y_4 - 2y_3 \\
&y_{3+2j}(0) = 0, \ \ y_{4+2j}(0) = s_j \ \ \ (j = 1, 2, 3, 4) \\
&1)\ y_5 = \mu_1 [y_7 + (\nu - 1 + k\varepsilon_0{}^{-1}) y_6] + \\
&\quad \mu_2 y_6 = y_9 = y_{10} = 0, \ r = 1 \\
&2)\ y_5 = \mu_1 [y_7 + (\nu - 1 + k\varepsilon_0{}^{-1}) y_6] + \mu_2 y_6 = y_{11} - \\
&\quad (1 + \nu) G_n y_9 = 0 \\
&\varepsilon_0{}^2 [y_{12} + (n - 2) y_{11} + (1 + \nu)(1 - n^2)(y_{10} + \\
&\quad (n - 1) y_9)] + \mu_1 (1 - y_1) y_6 = 0, \ r = 1 \\
&S_n y_i = (2n - 1)(y_i - r y_{i+1}) r^{-2}, \\
&G_n y_i = y_{i+1} - (n^2 - n + 1) y_i r^{-1}
\end{aligned} \tag{3.5}$$

The unknown parameters s_0, t_0, s_j are calculated by the method of alignment /8–10, 15/ by using the boundary conditions for $r = 1$. Analogously, the substitutions $x_s = r^{s-1} x_s{}^\circ$, $x_m = r^{m-1} x_m{}^\circ$; $B = r^{2s-1} B^\circ$, $F = r^{t-1} F^\circ$ are made to transform problem (2.11)–(2.14). The difference between the numerical results by the method of alignment presented below and the results from the programs in /4–6/ for a number $N = 100$ of mesh nodes does not exceed 0.03% for the calculation of p_0 and 1% for the calculation of the coefficients of system (2.7).

4. A spherical shell for a closed framed edge. Let q_0 be the classical value of the critical pressure for a complete sphere $p_0 = p_H q_0{}^{-1}$, $\Lambda = 2[3(1 - \nu^2)]^{1/4} (H/h)^{1/2}$, where p_H is the critical pressure of non-axisymmetric buckling of an ideal spherical shell. As in

/1–6/, we will set $\nu = 0.33$.

Using the formulas of Sect.3, we calculate for problems (1.1), 2) in (1.2) with $\Lambda = 6.6$, $\mu_1 = 0, \mu_2 = 1$ that two eigenvector-functions φ_1 and φ_2 with the harmonics $s = 2, m = 3$ correspond to the least eigenvalue $p_0 = 0.774$. To determine the critical loads $p_s(\xi d_1, \xi d_2)$ we derive the system of equations

$$\partial V/\partial \nu_k = 0, \quad \det\left\|\frac{\partial^2 V}{\partial \nu_k \partial \nu_j}\right\| = 0 \quad (k, j = 1, 2) \tag{4.1}$$

$$V = 194.4\nu_1^4 + 203.9\,\nu_2^4 + 789.8\,\nu_1^2\nu_2^2 + (p - p_0)(415\nu_1^2 + 666\,\nu_2^2) + \xi\,(d_1\nu_1 + d_2\nu_2)$$

Hence, following /5,6/, we have from (2.9) for critical pressures for $d_1 = 0$ and $d_2 \neq 0$:

$$p_2^{(i)} = p_0 - \eta_i\,(d_2\xi)^{2/3} \quad (i = 1, 2, 3, 4); \quad \eta_1 = 1.59; \quad \eta_2 = 1.6; \quad \eta_3 = \eta_4 = 8.64 \tag{4.2}$$

For the critical pressures we have for $d_1 \neq 0, d_2 = 0$

$$p_1^{(i)} = p_0 - \varkappa_i\,(d_1\xi)^{2/3} \quad (i = 5, 6, 7, 8); \quad \varkappa_5 = 1.85; \quad \varkappa_6 = 3.02; \quad \varkappa_7 = \varkappa_8 = 1.25 \tag{4.3}$$

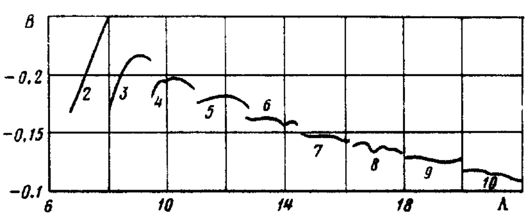
The arrangement of the cirtical load surfaces $p_s(\xi d_1, \xi d_2)$ is analogous to the case of a spherical shell with $\Lambda = 17$ (see /5,6/), where the greatest reduction in the critical value equals 0.41 for $R = |\xi|(d_1^2 + d_2^2)^{1/2} = 0.01$ and $\alpha = 1.37$, i.e., M_1 (1.37; 0.41). Therefore, for $R = 0.01$ the critical pressure of an imperfect shell is 53% less than p_0, and 11.1% less when taking account of buckling by one mode. The strong reduction in the bifurcation value of p_0 is characterized also by the relationship $(p_2^{(3)} - p_0)(p_2^{(1)} - p_0)^{-1} \approx 5.4$, where $p_2^{(1)}$ and $p_2^{(3)}$ are, respectively, the critical loads taking one and two eigen modes into account.

For $\Lambda = 9$ two eigenvector-functions φ_1 and φ_2 with $s = 4, m = 5$ correspond to the value $p_0 = 0.776$. The potential function has the form

$$V = 322.5\nu_1^4 + 384.6\nu_2^4 + 1447\nu_1^2\nu_2^2 + (p - p_0)(765.3\nu_1^2 + 1018\nu_2^2) + \xi\,(d_1\nu_1 + d_2\nu_2) \tag{4.4}$$

Here, (4.2) and (4.3) hold for $\eta_1 = 1.72$, $\eta_2 = 1.73$; $\eta_3 = \eta_4 = 3.7$, $\varkappa_5 = 1.79$, $\varkappa_6 = 2.05$, $\varkappa_7 = \varkappa_8 = 1.76$. The arrangement of the critical load surfaces is analogous to the case $\Lambda = 17$, where C_1 (0.62; 0.02), M_1 (1.18; 0.19). The critical pressure of the imperfect shell is 24% less than p_0 for $R = 0.01$, and 10.7% less than taking account of buckling in one mode. For $\Lambda = 9$ we note the experiment in /16/.

5. A sperhical shell with a free clamped edge. In this case calculations of the eigenvalues of the problem (1.4) for $n = 1$ together with the results in /6,9/ for $n \geq 2$ show that for $\nu = 0.3$ non-axisymmetric buckling of an ideal spherical shell is possible for $\Lambda \geq 6.3$. The graph of the coefficients $b(\Lambda) = -a_1 p_0^{-1}$, introduced in /3/ to estimate the shell response to imperfections for the least bifurcation value p_0, obtained by the application of the alignment and matrix factorization methods, is shown in the figure for the appropriate boundary value problem (1.1), and 1) in (1.2) for $\mu_1 = 1, \mu_2 = 0$. At the points $\Lambda =$ 6.7, 8.045, 9.52, 11.1, 12.75, 14.5, 16.3, 18, 20, 22 the eigenvalues p_0 belong to the double spectrum and equal respectively, 0.345, 0.348, 0.343, 0.335, 0.327, 0.318, 0.310, 0.302, 0.295, 0.288. Eigenfunctions with the numbers of the harmonics i and $i + 1$ correspond to the value of p_0 at the i-th place. Moreover, $p_0(6.3) = 0.331$. A graph agreeing with Fig.8 in /3/ is obtained for the coefficient b when checking the numerical programs of this paper in the case of rigid edge clamping.



In the case of buckling in two modes φ_1 and φ_2 with the harmonics $s = 2, m = 3$, for $\Lambda = 8.045$, $\nu = 0.33$ we obtain the following potential function for the least bifurcation value $p_0 = 0.348$

$$V = 57.1\nu_1^4 + 50.6\nu_2^4 + 244.6\nu_1^2\nu_2^2 + (p - p_0)(1323\nu_1^2 + 1700\nu_2^2) + \xi\,(d_1\nu_1 + d_2\nu_2)$$

Here $\eta_1 = \eta_2 = 0.737$, $\eta_3 = \eta_4 = 1.853$; $\varkappa_5 = 0.818$, $\varkappa_6 = 0.991$, $\varkappa_7 = \varkappa_8 = 0.652$, M_1 (1.18, 0.09), C_1 (0.62, 0.006). The critical pressure of an imperfect shell when $R = 0.01$ is 26.5% less than p_0 for an imperfect shell, and 9.8% less when taking account of buckling in just one mode.

Note that the graph in the figure does not agree with an analogous graph in Fig.3 in /17/, where the calculations were performed by the alignment method by means of (13), (14), (29) and (30) from /2/, where the change of variables

$\{x, \theta, \Phi\} = \lambda\{r, \beta, \psi\}, \{w_{1n}, f_{1n}, \rho, X\} = r^n\{\lambda W_n, \lambda F_n, r^n\tau, r^n\omega\}, \lambda = \Lambda$
is made in going over to the Cauchy problem. Such a substitution will obviously result in unsatisfactory results since, according to /17/, when integrating in the interval $[0, 1]$ the solutions of the above-mentioned boundary value problems are replaced by a segment of the Taylor series for $r \in [0, \Delta r]$, where Δr varies between the limits 0.11—0.27. For instance, it is asserted in /17/ on the basis of numerical computations that "for fixed n, b is a negative continuous function of λ". However, in the normalization of the eigenfunctions used in /2—6/, for $n = 2$ the correct computations give $b(8.76) = -8.32$ and $b(8.77) = 92.4$. Hence, at the very least a variability in the sign of the coefficient b results.

The sign change for the coefficient b in the interval $8.76 \leqslant \Lambda \leqslant 8.77$ indicates the passage of the mode with harmonic $n = 2$ from unstable into stable. A value Λ_* exists in this interval such that $p_0 \to 0.3557$, $L_{11} \to 2852$, $b = -L_{30}L_{11}^{-1}p_0^{-1} \to \mp\infty$ as $\Lambda \to \Lambda_* \mp 0$ and the amplitude v_1 of the appropriate non-axisymmetric mode satisfying the bifurcation equation $L_{30}v_1^3 + L_{11}(p - p_0)v_1 + \xi L_0 = 0$ tends to zero, i.e., for $\Lambda = \Lambda_*$ the solution "sits" on the axis $v_1 = 0$. This latter remark is due to V.A. Trenogin.

6. A spherical shell with a fixed hinge-supported edge. Numerical computations of problems (1.1), 2) in (1.2) for $\mu_1 = 1, \mu_2 = k = 0, \nu = 0{,}33$ show that the non-axisymmetric buckling of an ideal spherical shell can hold for $\Lambda \geqslant 3.4$. At the point $\Lambda = 4.36, 5.655, 6.94, 8.02, 9.35, 10.6, 11.95$ the eigenvalues p_0 are repeated and equal to, respectively, 0.652, 0.647, 0.655, 0.664, 0.669, 0.674, 0.679. The eigenfunctions with numbers i and $i+1$ of the harmonics correspond to the value of p_0 at the i-th place. In the case of buckling in two modes φ_1 and φ_2 with the harmonics $s = 2, m = 3$, for $\Lambda = 5.655; \nu = 0.33$ and $p_0 = 0.647$ we obtain the potential function

$$V = 234.7v_1^4 + 250.9v_2^4 + 988.9v_1^2v_2^2 + (p - p_0)(622.7v_1^2 + 913.9v_2^2) + \xi(d_1v_1 + d_2v_2)$$

Here

$$\eta_1 = 1.55;\ \eta_2 = 1.55;\ \eta_3 = \eta_4 = 5.21;\ \varkappa_5 = 1.72;\ \varkappa_6 = 2.28;\ \varkappa_7 = \varkappa_8 = 1.33;\ M_1(1.18; 0.25),\ C_1(0.51; 0.03).$$

The critical pressure of an imperfect shell is 39.1% less than p_0 for $R = 0{,}01$, and 12.3% less when taking account of buckling in one mode.

7. Asymptotic analysis as $\varepsilon_0 \to 0$. Let $\varepsilon_0 \to 0$ $(\Lambda \to \infty)$. Then in the case of the boundary conditions 2) in (1.2), for the axisymmetric solution $x^*(r)$ as $\varepsilon_0 \to 0$ the following asymptotic representations hold /13,14/

$$\begin{aligned}
&u \sim \varepsilon_0 g, \quad v \sim -2pr + \varepsilon_0 h, \quad p < 1 \qquad (7.1)\\
&g = s_1(s_2b^{-1}x_\varepsilon + y_\varepsilon), \quad h = s_1(2ab + pb^{-1}s_2)x_\varepsilon + s_1(p - 2as_2)y_\varepsilon\\
&2a^2 = 1 - p, \quad 2b^2 = 1 + p, \quad t = (1 - r)\varepsilon_0^{-1}\\
&x_\varepsilon = e^{-at}\sin bt, \quad y_\varepsilon = e^{-at}\cos bt\\
&s_1 = 2p(\nu - 1)(k + 2a)^{-1}\\
&s_2 = k + a, \quad \Lambda = \varepsilon_0^{-1}
\end{aligned}$$

Following /1,3,11/ and setting $r = 1 - \varepsilon_0 t, \sigma = n^2\varepsilon_0^2$, from (1.4) as $\varepsilon_0 \to 0$ we obtain

$$\begin{aligned}
&l_\sigma^{(1)}x_\sigma \equiv \omega'''' - 2\sigma\omega'' + \sigma^2\omega + \sigma\psi + 2p\omega'' - 2p\sigma\omega - \sigma h'\omega - \sigma g'\psi - \psi'' = 0 \qquad (7.2)\\
&l_\sigma^{(2)}x_\sigma \equiv -\psi'''' + 2\sigma\psi'' - \sigma^2\psi - \sigma g'\omega - \omega'' + \sigma\omega = 0, \quad x_\sigma \equiv (\omega, \psi)\\
&x_\sigma(\Lambda) = x_\sigma''(\Lambda) = 0, \quad [\omega = \mu_1(\omega'' - k\omega') + \mu_2\omega' = \psi'' + \sigma\nu\psi = \psi''' -\\
&\quad \sigma(2 + \nu)\psi' + \mu_1\omega']_{t=0} = 0, \quad (\ )' = \frac{d}{dt}(\ )
\end{aligned}$$

to determine the asymptotic value of the upper critical pressure of non-axisymmetric buckling p^a, the amplitudes of its eigenvector functions $(\gamma(t), \delta(t))$ and σ^a.

The surfaces of the asymptotic values of the critical loads are determined from the system of bifurcation equations (2.7), where the coefficients are determined by the formulas /11/

$$\begin{aligned}
&a_1 = b_2 = a_0 = c_0^{-1}\int_0^\Lambda\left(z_1\beta - z_2\alpha - \frac{1}{2}y_1\rho_1 + \frac{1}{2}y_2\rho_2\right)dt, \quad a_2 = b_1 = b_0 = \qquad (7.3)\\
&(4c_0)^{-1}\left\{4\int_0^\Lambda(\alpha z_2 - \beta z_1)\,dt - \sigma\int_0^\Lambda[\varkappa_1(\gamma'\delta + \gamma\delta') + \varkappa_2\gamma\gamma' + f_1(\gamma''\delta + \delta''\gamma -\right.\\
&\left.2\gamma'\delta') - f_2(\gamma'^2 - \gamma\gamma'')]\,dt\right\}, \quad c_0 = 2(\nu - 1)\alpha(0) + \frac{1}{2}\int_0^\Lambda\{\sigma\gamma^2 + \gamma'^2 - 4g\beta\}\,dt, \quad p = p^a, \quad \sigma = \sigma^a
\end{aligned}$$

$$d_i = d_i^\circ = -\sigma(4c_0)^{-1} q_{1i} \int_0^\Lambda \{(2p+h')\gamma + g'\delta\} t\,dt, \quad q_{1i} = \frac{\partial \zeta_i}{\partial r}\Big|_{r=1},$$
$$\zeta_i(1,\theta) = 0$$
$$m_i^2 \varepsilon_0^2 \to \sigma^a, \quad \zeta(r,\theta) = \varepsilon_0[\zeta_1(r)\cos m_1\theta + \zeta_2(r)\cos m_2\theta];$$
$$i = 1, 2$$

For a closed framed edge $\zeta_i(1,\theta) = q_{1i} = 0$ and the coefficients d_i^* and D_i from /11/ are zero. In this case we have for $\zeta = \sum_{i=1}^n \zeta_i(r)\cos m_i\theta$

$$d_i^* = (8c_0)^{-1}\sigma q_{2i} \int_0^\Lambda \{(2p+h')\gamma + g'\delta\} t^2\,dt, \quad q_{2i} = \frac{\partial^2 \zeta_i}{\partial r^2}\Big|_{r=1}$$

The formula for D_i in /11/ is therefore obtained by replacing $2p$ by $-k_0$.

Here $\alpha(t), \beta(t)$ and $\rho = (\rho_1, \rho_2)$ are determined from the boundary value problems

$$\beta'' + 2p\beta + \alpha = z_1, \ \alpha'' - \beta = z_2, \ z_1 = -\frac{\sigma}{2}(\gamma\delta)', \ z_2 = -\frac{\sigma}{2}\gamma\gamma', \ [\mu_1(\beta' - k\beta) + \tag{7.4}$$
$$\mu_2\beta = \alpha']_{t=0} = 0, \quad \alpha(\Lambda) = \beta(\Lambda) = 0$$
$$l_{4\sigma}^{(1)}\rho = y_1, \ l_{4\sigma}^{(2)}\rho = y_2, \ y_1 = -\frac{\sigma}{2}(\gamma''\delta + \gamma\delta'' - 2\gamma'\delta'), \ y_2 = -\frac{\sigma}{2}(\gamma''\gamma - \gamma'^2).$$
$$[\rho_1 = \mu_1(\rho_1'' - k\rho_1') + \mu_2\rho_1' = \rho_2'' + 4\sigma\nu\rho_2 = \rho_2''' - 4\sigma(2+\nu)\rho_2' + \mu_1\rho_1']_{t=0} = 0, \quad \rho(\Lambda) = \rho''(\Lambda) = 0$$

The operators $l_{4\sigma}^{(1)}, l_{4\sigma}^{(2)}$ are obtained, respectively, from expressions for the operators $l_\sigma^{(1)}, l_\sigma^{(2)}$ in (7.2) by replacing σ by 4σ. Note that in this notation, instead of $l_{2\sigma}^{(1)}, l_{2\sigma}^{(2)}$ in /11/, one should read $l_{4\sigma}^{(1)}, l_{4\sigma}^{(2)}$ respectively. For the functions $f = (f_1, f_2)$ and e_1, e_2 also concentrated in the boundary layer, we have

$$l_{4\sigma}^{(1)}f = \frac{1}{2}(s_1 + s_2), \quad l_{4\sigma}^{(2)}f = \frac{1}{2}(s_3 + s_4) \tag{7.5}$$
$$s_1 = -2\sigma(\gamma''\delta + \delta''\gamma), \quad s_2 = 4\sigma\delta'\gamma', \quad s_3 = -2\sigma\gamma''\gamma, \quad s_4 = 2\sigma\gamma'^2$$
$$[f_1 = \mu_1(f_1'' - kf_1') + \mu_2 f_1' = f_2'' + 4\sigma\nu f_2 =$$
$$f_2''' - 4\sigma(2+\nu)f_2' + \mu_1 f_1']_{t=0} = 0, \quad f(\Lambda) = f''(\Lambda) = 0$$
$$\varkappa_1'' - \varkappa_2 + 2p\varkappa_1 = \sigma(\gamma\delta)', \ \varkappa_2'' + \varkappa_1 = -\sigma\gamma\gamma'$$
$$\varkappa_1 = -e_1', \ \varkappa_2 = -e_2', \ \varkappa_1(\Lambda) = \varkappa_2(\Lambda) = 0$$
$$[\mu_1(\varkappa_1' - k\varkappa_1) + \mu_2\varkappa_1 = \varkappa_2']_{t=0} = 0$$

Formulas (7.1)–(7.5) are derived from (2.11)–(2.14) by the boundary layer method as $\varepsilon_0 \to 0$, and their extension to strictly convex shells under axisymmetric loading is performed by using the results obtained in /11,14/.

The boundary value problems (7.2), (7.4), (7.5) were solved as in /11/, where numerical results are presented for $\mu_1 = 0, \mu_2 = 1$. For a fixed hinge - supported edge ($\mu_1 = 1, \mu_2 = k = 0$) we have $p_0 \to p^a = 0{,}710, m_i^2\varepsilon_0^2 \to \sigma^a = 0.645\ (i = 1{,}2), a_0 = 0.7284, b_0 = 1.4568$. For the critical pressures we have (4.2) and (4.3) with p_0, d_1, d_2 replaced by $p^a, d_1^\circ, d_2^\circ, \eta_1 = \varkappa_5 = 1{,}700, \eta_2 = \varkappa_6 = 1{,}799, \eta_3 = \varkappa_7 = 2.452$. The values $\iota = 1{,}5$ are obtained when taking account of buckling in just one mode. The critical load surfaces have the same form as in /5/, where $M_1(5\pi/16, 0.132), C_1(\pi/4, 0{,}02)\ A_i = B_i$. Assuming that p^a is a hextuple eigenvalue, we arrive at formula (11) in /11/, where $\gamma_1^a = \eta_1$, $\gamma_2^a = \eta_3; \gamma_3^a = 6{,}048; \gamma_4^a = 9{,}511; \gamma_5^a = 12{,}943; \gamma_6^a = 16{,}362; \gamma_0^a = \eta_2$. These formulas explain the high sensitivity of very thin elastic shells to small initial deviations of their surfaces from ideal geometric shape.

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QUASI-TRANSVERSE SHOCK WAVES IN AN ELASTIC MEDIUM IN THE CASE OF SPECIAL TYPES OF INITIAL STRAIN*

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Weak shock waves in isotropic elastic medium under an arbitrarily small initial strain were considered in /1,2/. The present paper deals with shock waves with higher order of symmetry, when there are special types of initial strain; all the results are obtained in explicit form.

1. Formulation of the problem. The investigations are carried out using the same formulation and the same degree of accuracy as in /1—3/. The general form of the elastic potential of the isotropic medium is given by the expression

$$\Phi=\rho_0U(\varepsilon_{ij}, S)={}^1/_2\lambda I_1^2+\mu I_2+\beta I_1I_2+\gamma I_3+\delta I_1^3+\xi I_2^2+\rho_0T_0(S-S_0)+\text{const.}$$

$$I_1=\varepsilon_{ii},\quad I_2=\varepsilon_{ij}\varepsilon_{ij},\quad I_3=\varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}$$

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial w_i}{\partial\xi_j}+\frac{\partial w_j}{\partial\xi_i}+\frac{\partial w_k}{\partial\xi_i}\frac{\partial w_k}{\partial\xi_j}\right)$$

Here U is the internal energy, ρ_0 is the density in the stress-free state, T is the temperature, S is the entropy, ε_{ij} are the finite strain tensor components, w_i are the displacement tensor components, ξ_i are Lagrangian coordinates, and the coordinate system in the stress-free state is rectangular Cartesian. The axes of this system are chosen so that the

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